Elliptic Ruijsenaars-Schneider models on the complex projective space

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Abstract

We construct elliptic Ruijsenaars-Schneider models whose completed center-of-mass phase space is the complex projective space with the Fubini-Study symplectic form. For n particles, these models are labelled by an integer \( p \in \{1, \ldots, n-1\} \) relative to \( n \) and a coupling parameter \( y \) varying in a certain punctured interval around \( p/n \). Our work extends Ruijsenaars’ pioneering study of compactifications that imposed the restriction \( 0 < y < 1/n \), and also builds on an earlier derivation of such compactified models with trigonometric potential by Hamiltonian reduction. This is a joint work with László Feher.

Embedding the local phase space into \( \mathbb{CP}^{n-1} \)

We now introduce the map

\[ E: \mathbb{CP}^{n-1} \to \mathbb{C}^n, (x, \phi) \mapsto \omega = (\omega_1, \ldots, \omega_n) \]

with the complex coordinates having squared absolute values

\[ |\omega_j|^2 = \text{sgn}(M)(x_j - x_{j+p} - y), \quad j = 1, \ldots, n. \]

and arguments

\[ \text{arg}(\omega_j) = \text{sgn}(M)\left(\sum_{k=1}^{n-1} \Omega_k(\phi_{k+1} - \phi_k)\right), \quad j = 1, \ldots, n, \text{ arg}(\omega_n) = 0, \]

where \( \phi_i \equiv 0 \) and the \( \Omega_k \) \{k = 1, \ldots, n-1\} are integers chosen in such a way that

\[ E \left( \sum_{j=1}^{n} d\phi_j \right) = e^{i\omega}. \]

Proposition. The maps formed by the integers \( \Omega_k \) can be written as \( \Omega = B - C \), where \( B \) is a \((0,1)\)-matrix of size \((n-1) \times (n-1)\) with zeros along certain diagonals given by

\[ B_{\ell,k} = \begin{cases} 1, & \text{if } k = m + \ell (\text{mod } n) \text{ for some } \ell \in \{1, \ldots, n-1\}, \\ 0, & \text{otherwise}. \end{cases} \]

and \( C \) is also a binary matrix of size \((n-1) \times (n-1)\) with zeros along columns given by

\[ C_{\ell,k} = \begin{cases} 0, & \text{if } k \equiv \ell (\text{mod } n) \text{ for some } \ell \in \{1, \ldots, n-1\}, \\ 1, & \text{otherwise}. \end{cases} \]

We use the above map \( E \) to embed the local phase space \( P^{n-1} \) into the complex projective space \( \mathbb{CP}^{n-1} \), equipped with the rescaled Fubini-Study form \( |\omega|^2_M \). This embedding reads

\[ n_{\Omega}(\lambda) \in P_{\Omega}^{n-1} \to \mathbb{CP}^{n-1}, \]

where \( n_{\Omega}(\lambda) \) denotes the natural projection of the sphere \( S^{2n-1} \) to \( \mathbb{CP}^{n-1} \), i.e.

\[ n_{\Omega}(\lambda) = \sum_{j=1}^{n} d\phi_j \to \lambda = \sum_{j=1}^{n} d\phi_j. \]

\( n_{\Omega} \) is smooth, injective and its image is the open submanifold for which \( \sum_{j=1}^{n} |\omega_j| 
eq 0 \).

Extension of the Lax matrix

A spectral parameter dependent local Lax matrix of the model is given by

\[ L^\Omega_n(x, \phi) = (x, \phi) \]

\[ \left( \begin{array}{cccc} \text{arg}(\omega_1) & \text{arg}(\omega_2) & \cdots & \text{arg}(\omega_n) \\ \text{sgn}(M)(x_1 - x_2 - y) & \text{sgn}(M)(x_2 - x_3 - y) & \cdots & \text{sgn}(M)(x_{n-1} - x_n - y) \\ \text{sgn}(M)(x_1 - x_2 - y) & \text{sgn}(M)(x_2 - x_3 - y) & \cdots & \text{sgn}(M)(x_{n-1} - x_n - y) \\ \vdots & \vdots & \ddots & \vdots \\ \text{sgn}(M)(x_1 - x_2 - y) & \text{sgn}(M)(x_2 - x_3 - y) & \cdots & \text{sgn}(M)(x_{n-1} - x_n - y) \end{array} \right) \]

with the spectral parameter \( \lambda \) and the positive smooth functions

\[ V_j(x, \phi) = \text{sgn}(\omega_j) (\sum_{k=1}^{n} \text{arg}(\omega_k) - \text{arg}(\omega_j) + \text{sgn}(\lambda) \Delta_j), \quad j = 1, \ldots, n, \]

It can be shown that \( V_j(x, \phi) = |\omega_j|^2_M(x, \phi) \) and \( V_j(x, -\phi) = |\omega_j|^2_M(x, -\phi) \) with the functions \( V_j(x, \phi) \) \( V_j(x, -\phi) \) possessing smooth extensions to \( \mathbb{CP}^{n-1} \).

Theorem. The local Lax matrix \( L^\Omega_n(x, \phi) \) has a smooth global extension \( L^\Omega_n(x, \phi) \) to the complex projective space \( \mathbb{CP}^{n-1} \) such that it satisfies the following identity

\[ L^{\Omega_n}_n(x, \phi) \| E(x, \phi) \| = \Delta_0^{-1} L^{\Omega_n}_n(x, \phi) \| E(x, \phi) \|, \quad \| E(x, \phi) \| \in \mathbb{CP}^{n-1}, \]

where \( \Delta_0 = \text{deg}(\Delta_0, \ldots, \Delta_n) \) with \( \Delta_j = \exp\left(\sum_{k=1}^{n} \Omega_k(\phi_{k+1} - \phi_k)^2\right) \) \( j = 1, \ldots, n-1 \).

The explicit formula for the resulting global Lax matrix can be found in [1].

References


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