Elliptic Ruijsenaars-Schneider models on the complex projective space

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Abstract

We construct elliptic Ruijsenaars-Schneider models whose completed center-of-mass phase space is the complex projective space with the Fubini-Study symplectic form. For n particles, these models are labelled by an integer $p \in \{1, \dots, n-1\}$ relative prime to n and a coupling parameter y varying in a certain punctured interval around $p\pi/n$. Our work extends Ruijsenaars's pioneering study of compactifications that imposed the restriction $0 < y < \pi/n$, and also builds on an earlier derivation of such compactified models with trigonometric potential by Hamiltonian reduction. This is a joint work with László Fehér.

Compactified elliptic Ruijsenaars-Schneider model

This model describes n interacting particles moving in one spatial dimension. The dynamics is governed by the Hamiltonian

$$H(\boldsymbol{x}, \boldsymbol{\phi}) = \sum_{j=1}^{n} \cos(\phi_j) \sqrt{\prod_{k \neq j} \left[\mathrm{s}(y)^2 \big(\wp(y) - \wp(x_j - x_k) \big) \right]}.$$

Here $\boldsymbol{x} = (x_1, \dots, x_n)$ and $\boldsymbol{\phi} = (\phi_1, \dots, \phi_n)$ collect the generalised coordinates and momenta of the particles, while y is a real parameter responsible for the strength of the interaction. The potential contains the Weierstrass $\wp$ function with half-periods $(\omega, \omega') \in \mathbb{R}_{>0} \times \mathrm{i}\mathbb{R}_{>0}$ and the 2ω-antiperiodic odd function s defined using the Weierstrass σ and ζ functions as

$$\mathrm{s}(x) = \sigma(x) \exp(-\zeta(\omega) x^2 / 2\omega).$$

Without loss of generality, we can set $\omega = \pi/2$. Then the Hamiltonian H is π-periodic in the parameter y. Since $y \to 0$ yields free particles, we can assume that

$$0 < y < \pi.$$

Remarks. (1) The model introduced by Ruijsenaars [3] has a slightly different Hamiltonian. Namely, the square root of each factor is taken in the products appearing in H.
(2) Sending $\omega' \to \mathrm{i}\infty$ gives rise to the compactified trigonometric RS model.

The local phase space

Fehér and Kluck [2] showed that the phase space of the (trigonometric) model can be one of only two drastically different forms depending on the parameter y. These two types of parameters form disjoint open subintervals that partition $(0, \pi)$. See the figure below.

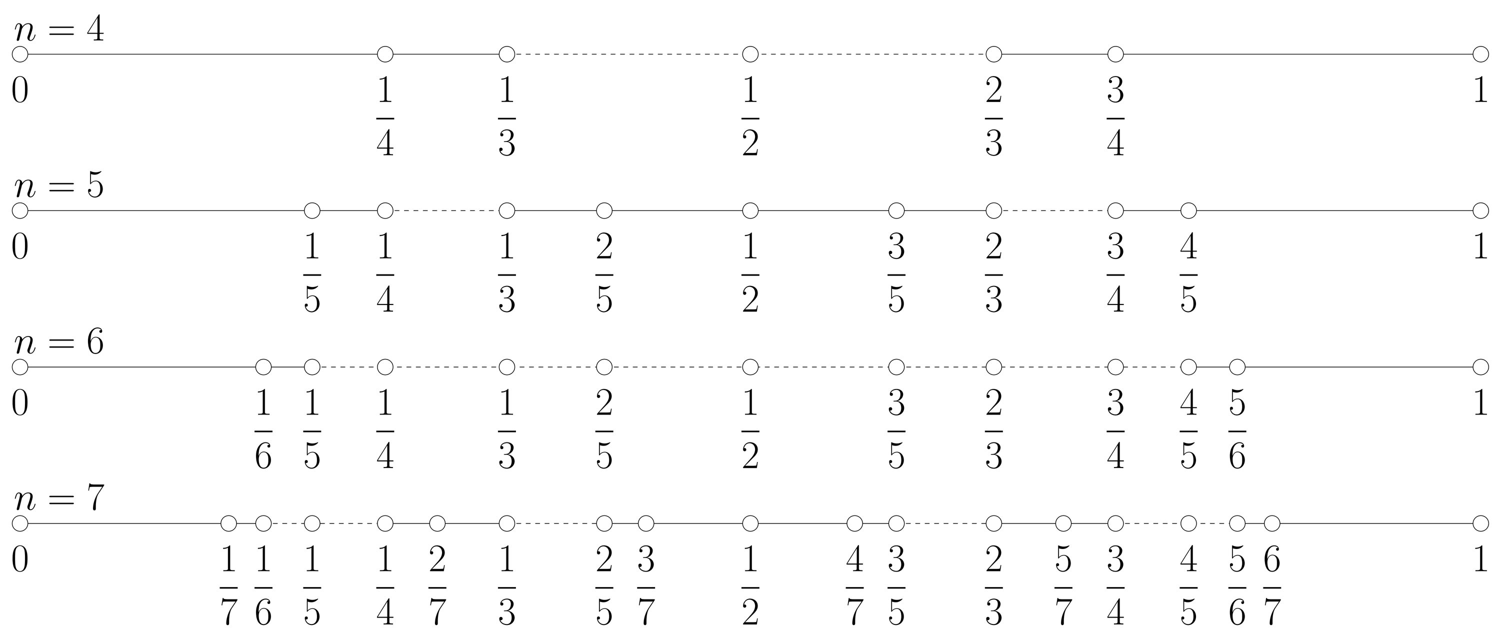

Figure. The range of y/π for $n = 4, 5, 6, 7$. The numbers displayed are excluded values. Admissible values of y form intervals of type (i) (solid) and type (ii) (dashed) couplings.

Here we only consider **type (i) parameters**, which can be characterised as follows. For a fixed integer $n \geq 2$, choose $p \in \{1, \dots, n-1\}$ to be a coprime to n, i.e., $\gcd(n, p) = 1$, and let q denote the multiplicative inverse of p in the ring $\mathbb{Z}_n$, that is $pq \equiv 1 \pmod{n}$. Then the parameter y can take its values according to either

$$\left(\frac{p}{n} - \frac{1}{nq} \right) \pi < y < \frac{p\pi}{n} \quad \text{or} \quad \frac{p\pi}{n} < y < \left(\frac{p}{n} + \frac{1}{(n-q)n} \right) \pi.$$

For such a type (i) parameter y, the local configuration space is the interior of a simplex in the center-of-mass hyperplane $E = \{ \boldsymbol{x} \in \mathbb{R}^n \mid x_1 + \dots + x_n = 0 \}$. Consider the parameter

$$M = p\pi - ny,$$

and note that $M > 0$ and $M < 0$ corresponds to y less/greater than $p\pi/n$, respectively. Then the local configuration space is given by

$$\Sigma_y = \{ \boldsymbol{x} \in E \mid \mathrm{sgn}(M)(x_j - x_{j+p} - y) > 0, \ j = 1, \dots, n \},$$

where we extended the indices in a periodic manner, that is $x_{n+k} = x_k - \pi$ for all k. The local phase space is the symplectic manifold

$$P_y^{\mathrm{loc}} = \{ (\boldsymbol{x}, e^{\mathrm{i}\boldsymbol{\phi}}) \mid \boldsymbol{x} \in \Sigma_y, \ e^{\mathrm{i}\boldsymbol{\phi}} \in \mathbb{T}^{n-1} \}, \qquad \omega^{\mathrm{loc}} = \sum_{j=1}^{n} dx_j \wedge d\phi_j.$$

where $\mathbb{T}^{n-1}$ is the $(n-1)$-torus in E.

Embedding the local phase space into $\mathbb{CP}^{n-1}$

We now introduce the map

$$\mathcal{E} \colon P_y^{\mathrm{loc}} \to \mathbb{C}^n, \quad (\boldsymbol{x}, e^{\mathrm{i}\boldsymbol{\phi}}) \mapsto \boldsymbol{u} = (u_1, \dots, u_n)$$

with the complex coordinates having squared absolute values

$$|u_j|^2 = \mathrm{sgn}(M)(x_j - x_{j+p} - y), \quad j = 1, \dots, n,$$

and arguments

$$\arg(u_j) = \mathrm{sgn}(M) \sum_{k=1}^{n-1} \Omega_{j,k} (\phi_{k-1} - \phi_k), \quad j = 1, \dots, n-1, \qquad \arg(u_n) = 0,$$

where $\phi_0 \equiv 0$ and the $\Omega_{j,k}$ $(j, k = 1, \dots, n-1)$ are integers chosen in such a way that

$$\mathcal{E}^* \left(\mathrm{i} \sum_{j=1}^{n} d\bar{u}_j \wedge du_j \right) = \omega^{\mathrm{loc}}.$$

Proposition. *The matrix formed by the integers $\Omega_{j,k}$ can be written as $\Omega = B - C$, where B is a $(0,1)$-matrix of size $(n-1)$ with zeros along certain diagonals given by*

$$B_{m,k} = \begin{cases} 0, & \text{if } k - m \equiv \ell p \pmod{n} \text{ for some } \ell \in \{1, \dots, n-q\}, \\ 1, & \text{otherwise,} \end{cases}$$

and C is also a binary matrix of size $(n-1)$ with zeros along columns given by

$$C_{m,k} = \begin{cases} 0, & \text{if } k \equiv \ell p \pmod{n} \text{ for some } \ell \in \{1, \dots, n-q\}, \\ 1, & \text{otherwise.} \end{cases}$$

We use the above map $\mathcal{E}$ to embed the local phase space P_y^{loc} into the complex projective space $\mathbb{CP}^{n-1}$ equipped with the rescaled Fubini–Study form $|M|\omega_{\mathrm{FS}}$. This embedding reads

$$\pi_{|M|} \circ \mathcal{E} \colon P_y^{\mathrm{loc}} \to \mathbb{CP}^{n-1},$$

where $\pi_{|M|}$ denotes the natural projection of the sphere $S^{2n-1}_{|M|}$ to $\mathbb{CP}^{n-1} \simeq S^{2n-1}_{|M|}/\mathrm{U}(1)$, i.e.

$$\pi^*_{|M|}(|M|\omega_{\mathrm{FS}}) = \mathrm{i} \sum_{j=1}^{n} d\bar{u}_j \wedge du_j.$$

$\pi_{|M|} \circ \mathcal{E}$ is smooth, injective and its image is the open submanifold for which $\prod_{j=1}^{n} u_j \neq 0$.

Extension of the Lax matrix

A spectral parameter dependent local Lax matrix of the model is given by

$$L_y^{\mathrm{loc}}(\boldsymbol{x}, e^{\mathrm{i}\boldsymbol{\phi}} | \lambda)_{j,k} = \frac{\mathrm{s}(y)\mathrm{s}(x_j - x_k + \lambda)}{\mathrm{s}(\lambda)\mathrm{s}(x_j - x_k + y)} [V_j(\boldsymbol{x}, y)]^{1/2} [V_k(\boldsymbol{x}, -y)]^{1/2} e^{\mathrm{i}\phi_k}, \quad \forall (\boldsymbol{x}, e^{\mathrm{i}\boldsymbol{\phi}}) \in P_y^{\mathrm{loc}},$$

with the spectral parameter λ and the positive smooth functions

$$V_j(\boldsymbol{x}, \pm y) = \mathrm{sgn}(\mathrm{s}(ny)) \prod_{k \neq j} \frac{\mathrm{s}(x_j - x_k \pm y)}{\mathrm{s}(x_j - x_k)}.$$

It can be shown that $V_j(\boldsymbol{x}, y) = |u_j|^2 W_j(\boldsymbol{x}, y)$ and $V_k(\boldsymbol{x}, -y) = |u_{k-p}|^2 W_k(\boldsymbol{x}, -y)$ with the functions $W_j(\boldsymbol{x}(\boldsymbol{u}), y), W_k(\boldsymbol{x}(\boldsymbol{u}), -y)$ possessing smooth extensions to $\mathbb{CP}^{n-1}$.

Theorem. *The local Lax matrix $L_y^{\mathrm{loc}}(\boldsymbol{x}, e^{\mathrm{i}\boldsymbol{\phi}} | \lambda)$ has a smooth global extension $L^{y,\pm}(\pi_{|M|}(\boldsymbol{u}) | \lambda)$ to the complex projective space $\mathbb{CP}^{n-1}$ such that it satisfies the following identity*

$$L^{y,\pm}((\pi_{|M|} \circ \mathcal{E})(\boldsymbol{x}, e^{\mathrm{i}\boldsymbol{\phi}}) | \lambda) = \Delta(\boldsymbol{\phi})^{-1} L_y^{\mathrm{loc}}(\boldsymbol{x}, e^{\mathrm{i}\boldsymbol{\phi}} | \lambda) \Delta(\boldsymbol{\phi}), \quad \forall (\boldsymbol{x}, e^{\mathrm{i}\boldsymbol{\phi}}) \in P_y^{\mathrm{loc}},$$

where $\Delta(\boldsymbol{\phi}) = \mathrm{diag}(\Delta_1, \dots, \Delta_n)$ with $\Delta_j = \exp\left(\mathrm{i} \sum_{k=1}^{n-1} \Omega_{j,k} (\phi_{k-1} - \phi_k) \right)$, $j = 1, \dots, n-1$, $\Delta_n = 1$.

The explicit formula for the resulting global Lax matrix can be found in [1].

References

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