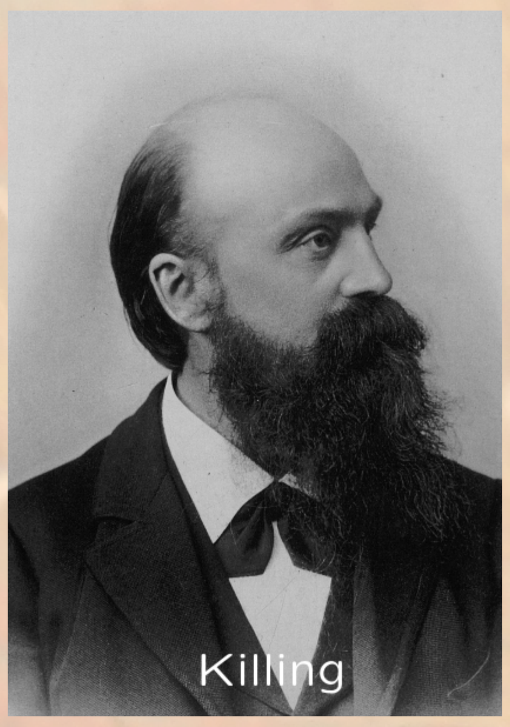
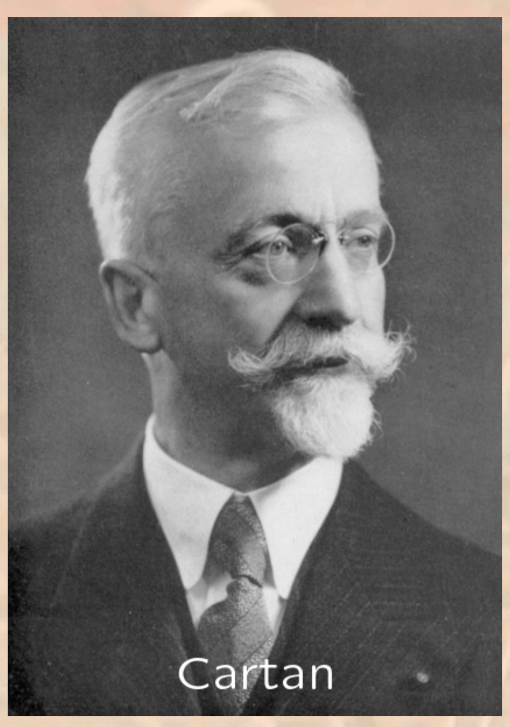
EXCE$_6$PTIONALLY BE$_7$AUTIFUL SYMME$_8$TRIES

TAMÁS F$_4$ERENC G$_2$ÖRBE

A bit of history



The classification of (semi)simple Lie algebras over the field of complex numbers is regarded by many as a jewel of mathematics. It was first described by German mathematician Wilhelm Killing in a series of papers published between 1888-1890. A more rigorous proof (and the case of real Lie algebras) was presented by Élie Cartan in his 1894 PhD thesis. In 1947 the 22-year old Eugene Dynkin worked out a modern, streamlined proof of the classification theorem. The theorem states that every semisimple complex Lie algebra is a "sum of building blocks", most of which belong to one of four infinite families. These are denoted by A_n, B_n, C_n, D_n, where n is an arbitrary positive integer. Surprisingly, there exist five exceptional "building blocks" that don't fit into the above families. They are named E_6, E_7, E_8, F_4, G_2.



The classification theorem heavily relies on abstract mathematical objects called root systems, which are symmetric configurations of vectors (usually) sitting in higher dimensional space. The dimension of this space is indicated by the subscripts, e.g. E_8 lives in an 8-dimensional Euclidean space.

Image credit: [K] ULB Münster. [C] Obituary Notices of Fellows of the Royal Society, **8**(21) 71-95, 1952. [D] The Eugene Dynkin Collection, Cornell University

Root system E_6

Number of root vectors: 72

Number of symmetries: 51,840

Root system E_7

Number of root vectors: 126

Number of symmetries: 2,903,040

Root system E_8

Number of root vectors: 240

Number of symmetries: 696,729,600

Kissing number in 8 dimensions: 240

Root system F_4

Number of root vectors: 48

Number of symmetries: 1152

Kissing number in 4 dimensions: 24

Root system G_2

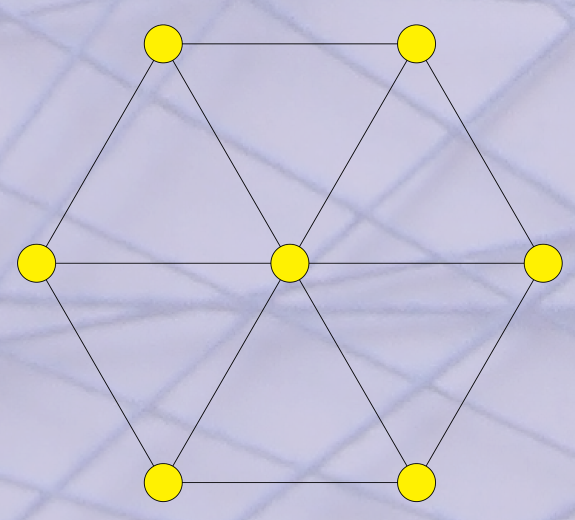
Number of root vectors: 12

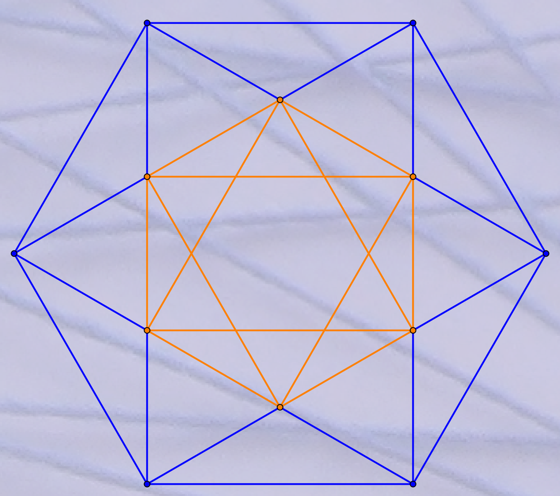
Number of symmetries: 12

Kissing number in 2 dimensions: 6

How it's made

The frame of a cube can cast shadows of different shapes, but only a few orientations lead to the most symmetric shadows, namely the projections through one of the four body diagonals. See the image on the right. The method of finding the "most symmetric shadows" can be generalized to higher dimensions, and this is how the string arts of the *exceptional root systems* E_6, E_7, E_8, F_4, G_2 were made. The needles of the string arts point where the end points of root vectors land after projection.





The connections are obtained by connecting every vector with their nearest neighbours before the projection.† A fun fact is that due to the left-right symmetry of the connections we have an even number of threads meeting at each needle. In graph theory such a structure is called an *Eulerian Circuit*. meaning that the connections with different colours can be drawn using a single, but really long piece of thread without cutting.‡

† There are some extra connections in the case of G_2.

‡ There's only one exception. Can you find it?